\noindent

Evaluation of the data Shows that in general refinement based on displacement and stress consistency results in well refined meshes although depending on the threshold this can be highly costly to compute. By contrast heuristics have shown to retain low computational overhead although volatile when used to refine meshes. Where edges are well specified for the heuristics quality can be seen to increase more rapidly than refinement based on displacement data. Heuristic refinement slows down with subsequent iterations which is likely a result of the regions yielding improvement with refinement becoming increasingly small and thus increasing the distance between the edges originally specified and the target meshing location. \\

\noindent

Heuristics therefore appear preferable in cases where they can be confirmed as good and where very high accuracy is not as important as reduced computational overhead. By contrast displacement refinement works work well when high accuracy is needed although incurs a higher computational overhead per gain in accuracy early on. \\

\noindent

An ideal hybrid configuration should therefore exhibit the following two characteristics:\\ \\

\noindent

\textbf{Edge Specification Assessment:} Establish quickly if the edges specified are any good when meshing, if they aren't stop continued use and switch to displacement refinement. \\ \\

\noindent

\textbf{Heuristic Refinement Halting:} If the heuristic refinement does appear to be working well establish at what point it is beginning to become inferior and then switch methods. \\

%%%%%%%%%%%%%%%%%%%%%%%%%

% a series of hybrid weightings were then run

\noindent

To evaluate each hybrid weighting the improvement results were averaged so as to provide a general indication given a variety of edge qualities used hybrid weightings with variable edge quality used for the heuristic method. The results for this can be seen below in figure x. \\

\begin{figure}[!h]

\centerline{**\includegraphics**[width=110mm, scale=1]{../Graphics/Graphs/BridgeCrossLoadingElementCount.png}}

\caption{Increase in element count for the various refinement methods over forty iterations}

**\label{fig:sub1}**

\end{figure}

\noindent

Looking at the numerical data in figure x the trend for heuristics to perform well initially before before losing ground to the displacement refinement process persists. This trend is most noticeable in the data for those hybrid processes which place a greater weighting upon upon the heuristic component. Although Heuristic methods can be seen to regain an increasing rate of improvement after the initial decline from the first iteration, this is still slower than that of the displacement method. Considering these results the most desirable hybrid probably will want to continue to utilise both approaches for the entire duration of its refinement process although will want to increasingly shift weighting in favour of displacement or stress refinement in favour of the heuristic. This does rule out the possibility of non traditional approaches in general however but simply indicates weaknesses in the current heuristic system being used.

\noindent

Running the same weightings for two additional models reveals consistency in these results . which suggest that given the design of the heuristic meshing process this trend can be generalised to describe the results for arbitrary structures regardless of variation in the edge quality.

\noindent %As a result of both methods running sequentially when combined as part of the system design there was little difference in the results of each method when run as part of a hybrid process when compared to its individual execution. Consequently the results for weighting using a hybrid approach were largely predicable using the results from execution of the individual methods, with a continuing trend of improvement increasing fastest in with those weightings that favour a good set of heuristic input witht the rate of improvement increasingly slowing in favour of refinement in areas of high displacement.

%effecting the mesh for the other method on subsequent iteration

the only observed effect when executing hybrid approaches was the

**\subsubsection{Hybrid Evaluation}**

It's clear looking at the data that a hybrid approach is potentially beneficial in scenarios where.

**\subsubsection{Analysis of and Addition of Metric(s)}**

\noindent

When looking at the initial results of the quality metrics for both the various simulations there were several key observations which greatly informed the remaining evaluation to be conducted. The maximum internal angles to elements are one good indication of general quality for the mesh and indeed gradual improvement with additional refinement iterations increasingly subdividing skewed elements into more uniform ones. Figure 11 below shows greater uniformity towards the optimum of 90 degrees for elements of type Quad4 is achieved for all but the poorest edge specification. Rate of improvement is fast for the first few iterations but then gradually declines. \\

Looking at the Figure its also clear that as a result of the heuristics being described in terms of edges

Increase in element count for the various refinement methods over forty iterations

\noindent

Although the Dittmer metrics providing a good indication of general improvement in quality across a mesh or a particular element in the majority of cases they gave little insight into the strengths of a particular element arrangement within a given mesh. Consequently it was possible to conclude using Dittmers metrics that the general quality of the resultant meshes was good with this quality being at lest retained and often improved, but not that the method refining the mesh had made a better selection of elements to mesh. \\

\cite{ElemQualAndChecks}. \\

\noindent

A more conclusive evaluation metric was clearly needed. Designing such a metric required clarity in the understanding that each method should provide as much information as possible to an engineer while retaining the lowest computational overhead possible, essentially how much additional information is revealed through each additional iteration of a given approach. It can also assume in this case that the concentration of displacement and therefore stress is what we are interested in finding. I concluded that the best means of evaluating each method was was to cross reference the resultant stress data for each iteration against the newly refined regions generated by each approach, the overall stress revealed within the output as a result of that part of the mesh being refined could then be summed and divided by the number of nodes created in order to reveal it through evaluation. This new metric became the average displacement revealed metric as can be seen in Figure 9b. \\

%\begin{figure}{\textwidth}

% \centering

% \includegraphics[width=0.9\linewidth]{../Graphics/Graphs/BridgeCrossLoadingElementCount.png}

% \caption{Execution time in seconds for each method to reach 40 refinement iterations}

% \label{fig:sub1}

%\end{figure}

%see appendix x figure for example

\noindent

%Currently the default setting for the evaluation function is to refine any elements with a stress greater than the 94th percentile for the model. The threshold setting is relatively arbitrary although choosing to mesh elements rated above the 94th percentile allowed the stress refinement method to be fairly compared against the heuristic methods since these typically only selected about 6\% of elements for refinement although this can clearly vary depending on the model and edges used.

% of input configurations was particularly important when analysing the heuristic method since by what edges were provided to it by the operator.

%When testing the stress refinement method the main approach altered

In order to evaluate general consistency in these results the experiments were rerun for

Results for which can be seen in

\newpage

**\subsection{Evaluation of Subsystems based on model evaluation}**

Running the system for the above models gave clear indications as to the overall effectiveness of both methods

Having run the system on the range of models described above it was possible to begin assessing it's ability to mesh and the use of Dittmers metrics for assessing

the quality of each mesh.

**\subsubsection{Analysis of Methods Using Metrics}**

When evaluating Dittmers metrics for distinguishing between meshes of different qualities there were several key observations that resulted in re assessment of how to evaluate the mesh qualities in general.

\iffalse

%graphs of Dittmer metrics showing little variation in the results

\begin{figure}[!h]

\centering

\begin{subfigure}{.5\textwidth}

\centering

**\includegraphics**[width=0.9\linewidth]{g}

\caption{Cylinder model used by Dolsak with edges labeled, each labelling corresponds to a rule generated by the Golem algorithm \cite{DolsakPaper91}}

**\label{fig:sub1}**

\end{subfigure}%

\begin{subfigure}{.5\textwidth}

\centering

**\includegraphics**[width=0.9\linewidth]{}

\caption{Diagram representing the general structure of the system with interactions between non internal entities such as files and LIA shown using dashed lines}

**\label{fig:sub2}**

\end{subfigure}

**\label{fig:test}**

\end{figure}

\fi

\noindent

Having coded this into the MeshQualityAssessment class the results for this metric could be plotted to evaluate each approach over a number of iterations for the different simulations. I also varied the selection of edges provided to heuristic refinement method which highlighted just how dependent the heuristic approach was on the quality of user expertise.

\begin{figure}[!h]

\centering

\begin{subfigure}{.5\textwidth}

\centering

**\includegraphics**[width=1.5\linewidth]{../Graphics/Graphs/BridgeCrossLoadingAverageDisplacementRevealed.png}

\caption{Bridge Cross Loading -Average Displacement Revealed Through Creation of Node per Iteration the plateau experienced by Stress Remeshing above 94th percentile is a consequence of all subsequent meshing being focused on an area already very highly meshed, in other words}

**\label{fig:sub1}**

\end{subfigure}%

**\label{fig:test}**

\end{figure}

%Subsequent research suggested

%Demonstrating potential for both heuristic and hybrid refinement for small models suggests that there is still a large amount of work to be done in this area. Although extensive work has been done by engineers and mathematicians to improve the numerical methods associated with the method for many years the improvements obtained

**\subsubsection{Evaluation of Stress Refinement Effectiveness}**

Comparisons across multiple models suggests that for general purpose scenarios where results of the simulation are highly unpredictable and the geometry irregular stress based refinement will continue to provide engineers with reliable results within a predictable although potentially costly amount of time. This scenario likely encapsulates the majority of use cases for FE analysis in the real world and therefore is likely to remain the prominent approach by practitioners.

It is clear however that the decline in performance is dependent upon the threshold used to detemine meshing

**\subsubsection{Evaluation of Heuristic Refinement Effectiveness}**

The most crucial observation which can be made through analysis of the simulation data is that heuristic equation based on model edges has resulting mesh quality has high variability. Despite this it is also clear that despite high variation in results for different edge specifications the general rate of growth or decline for the resultant metric values remains consistent. In addition to supporting the general case for using heuristics in order to aid the mesh refinement the success of Dolsaks published rules indicate that Muggletons underlying ILP approach for generation of the rules has some underlying merit. \\